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COMPUTATION OF THE AZIMUTH OF THE GREAT CIRCLE IN CARTESIAN COORDINATES

ABSTRACT

The commonly known method of calculations connected with the Great Circle sailing and with similar tasks is executed by navigators with geographical coordinates. This paper deals with the possibility of azimuth calculations of the direction between two points on the Earth surface with 3-D Cartesian coordinate system.

Keywords:
asimuth, Great Circle, Carthesian coordinates.

INTRODUCTION

From the daybreak of the navigation (in astronomy and geodesy as well) geographical coordinates are commonly used for the description of the position of the point in relation to the Earth. So it is natural that all computational methods are based on this coordinate system. It is especially perceptible with reference to calculations of the azimuth between points and distances between them, which in fact constitutes a basis of the planning in the navigation. However, calculations on the sphere or the spheroid with the use of angular coordinates are rather complicated. For curved or more complicated surfaces the metric can be used to compute the distance between two points by integration. Therefore many facilities were introduced, as for example logarithms and different sorts of tables. Different map projections have also been introduced what permitted to dissolve suchlike problems with geometrical methods. In spite of the development of the computer technique this problem still matters, and the evidence of that can be the publications [Pallikaris A. and Latsatos G., 2009] or [Earle M. A., 2008].

The introduction to the everyday use of satellite navigation systems and Inertial Navigation Systems, puts a problem in a new light, because these systems work in the three-dimensional space. Therefore, bringing the solution in form of position coordinates to the navigator demands additional calculations, which allow to present the three-dimensional calculation as two-dimensional on the surface of the ellipsoid. It is known, anyway, that algorithms of the calculation of coordinates in these devices work better basing on the system of spatial Cartesian coordinates than in geographical coordinates. In consequence, there can be raised a question, whether in the era of the general usage of computers the execution of basic calculations in navigation will be easier, if we will implement it in Cartesian coordinate system?

An aim of the article is the discussion of the certain method of the calculation of the azimuth between two points on the earth surface with the utilization of 3-D Cartesian coordinates.

GEOMETRY OF NORMAL SECTION OF THE ELLIPSOID IN CARTESIAN COORDINATES

The problem of calculation of direction and distance between two points appears mostly at the stage of travel planning, but it is also important in dead reckoning navigation. In that case, the problem means the calculation of the length and the azimuth of the geodesic line. This problem arises also when calculating the position with analytic methods, especially when we use such methods as least squares, Kalman Filter etc. The calculation of the geodesic line is comparatively complicated and labour-consuming, thereby universally complies different simplification.

In the present work we will use the often applied approach, which is the replacement of the geodesic line with the arc of the normal section of the ellipsoid. It is admissible because the difference between the length of the normal section of the ellipsoid and the length of the geodesic line is in most cases so small that it can be skipped. This differences are expressed by equation [Morozov, 1969]:

$$S_P - S_G = \frac{e^4}{360a^4} S_G^5 \cos^4 \varphi_1 \sin^2(2Az)\,, \tag{1}$$

where:

S_P — the length of the arc of the normal section;
S_G — the lenght of the geodesic line;
Az — azimuth;
e^2 — first eccentrity of the ellipsoid;
a — semimajor axis of the ellipsoid;
φ — latitude of the start point.

This difference is the biggest on the equator and in case of $Az = 45^{\circ}$. However, in case of distance 1000 km it reaches only 0,00007 m. Even in the geodesy such values are acceptable as very small, in comparison of accidental errors occurring in measurements of angles and distances. Taking into account that in the navigation both requirements and measuring-possibilities do not exceed the geodesic ones, the replacement of the geodesic line by the arc of the normal section seems fully legitimated.

Algorithms of solution of this assignment basing on the theory of normal sections are known for example from publications of [Jordan and other, 1958] or [Zakatov, 1959]. However authors of these formulas still use geodesic coordinates, and consequently — trigonometric functions. In present work only 3-D Cartesian coordinates are applied, thanks to which the calculation can be deduced from classical methods of the geometry of the plane in the three-dimensional space.

Let us consider the plane with two points P_1 and P_2 (Fig. 1) situated on the surface of the reference ellipsoid. Additionally let us consider the rectangular Cartesian coordinate system with the *X* axis parallel to the conventional zero meridian of Greenwich, the *Y* axis towards the east, and *Z* axis parallel to the Earth rotation axis, to which the minor axis of reference ellipsoid is also parallel. If mentioned crossing plane is normal to the reference ellipsoid in the point P_1, then (apart from the special cases when point P_1 is situated on the Pole or on the Equator) it does not pass through the origin of the coordinates system. It crosses Z-axis in the point *k* whose coordinates are described as: $(0, 0, z_k)$.

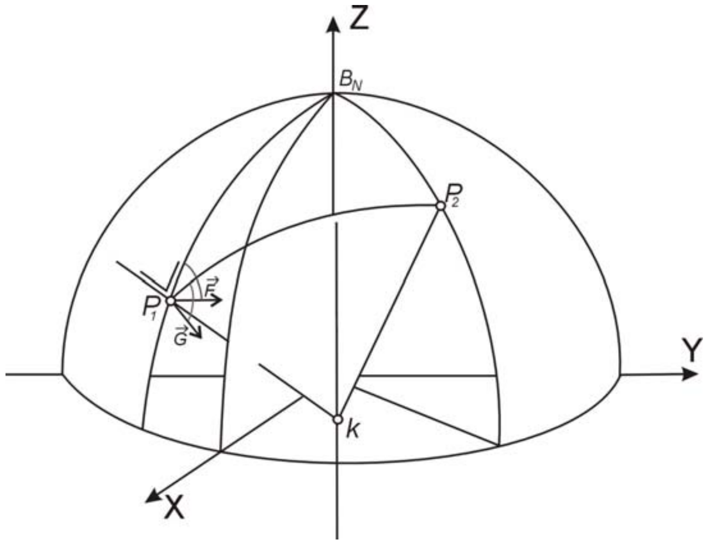

Fig. 1. The normal section of the ellipsoid and normal vectors of the main surfaces [own study]

$$z_k = -e^2 N_1 \sin(\varphi_1) \tag{2}$$

Let us present the equation (2) in a different form, aiming to replace of geodesic coordinates by Cartesian one. As the coordinate z can be described with the equation:

$$z = [N(1-e^2) + H]\sin(\varphi), \tag{3}$$

where:
N — radius of curvature in E-W direction;
H — high of the point abowe ellipsoid.

Than

$$N\sin(B) = \frac{z}{1-e^2} - H\frac{\sin(\varphi)}{1-e^2}. \tag{4}$$

Putting (4) into (2) we receive:

$$z_k = -\frac{e^2}{1-e^2}z + \frac{e^2}{1-e^2}H\sin(\varphi). \tag{5}$$

However

$$\frac{e^2}{1-e^2} = e^{'2}, \tag{6}$$

so finally equation (2) assumes form:

$$z_k = -e^{'2}z + e^{'2}H\sin(\varphi). \tag{7}$$

The first element of the sum (7) is a linear function and its value changes from 0 to 43 km, while the second one — assuming that value H really occurs at the seas (±200m) — changes from 0 to approx. 1,5 m. On this base we will found that this is the value so small that we will simplify (7) to the form:

$$z_k = -e^{'2}z. \tag{8}$$

AZIMUTH OF NORMAL SECTION OF THE ELLIPSOID

The azimuth of the normal section of the ellipsoid (*Az*) is the angle, which is measured in the normal plane to the ellipsoid in the point P_1 (that is to say in the surface of the horizon). This azimuth is laying between the plane of the meridian and the plane of the normal section (Fig. 1.). If both surfaces (the normal section and the meridian) are described with their normal vectors $\vec{F}$ and $\vec{G}$, the value of the azimuth can be calculated as the angle between these vectors.

The above-argumentation contains the certain simplifications. In particular we should take into account the influence of the height of both points over the ellipsoid on the azimuth, because aiming on point P_2 usually we make the measurement not in the surface of the horizon of the point P_1, so not with relation to the direction of the plumb-line in this point. This causes errors in calculations. In similar way the deviation of the plumb-line influences on the results, however it can be skipped. Taking into account values of deviations of the plumb-line on oceans and distances from the reference ellipsoid, we reach a conclusion that this error will not exceed the part of a second.

In case that the plane of the meridian of the point P_1 passes also by the pole and the centre of the ellipsoid, this plane is described by following coordinates:

$$P_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}; \qquad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

If so, the plane of the meridian can be described by following equation:

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = xy_1b - yx_1b = 0. \tag{10}$$

The plane of the normal section will pass by points P_1, P_2, and k:

$$P_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}; \qquad k = \begin{bmatrix} 0 \\ 0 \\ -z_1e'^2 \end{bmatrix} \tag{11}$$

and we will describe it with the equation (12):

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ 0 & 0 & -z_1 e^{'2} & 1 \end{bmatrix} = 0 . \tag{12}$$

We will mark coefficients of equations which describes both planes adequately:

F_i — coefficients of the equation which describes the surface of the meridian;
Gi — coefficients of the equation which describes the plane of the normal section;
$i = 1, 2, 3, 4$.

Taking it into account we receive the following formulae:

$$xF_1 + yF_2 + zF_3 + F_4 = 0 , \tag{13}$$

where:
$F_1 = y_1 b$;
$F_2 = -x_1 b$;
$F_3 = 0$;
$F_4 = 0$.

By analogy:

$$xG_1 + yG_2 + zG_3 + G_4 = 0 , \tag{14}$$

where:

$$G_1 = z_1 \left[y_1 e^{'2} - y_2 \left(1 + e^{'2} \right) \right] + z_2 y_1 ,$$

$$G_2 = z_1 \left[x_2 \left(1 + e^{'2} \right) - x_1 e^{'2} \right] - z_2 x_1 ,$$

$$G_3 = x_1 y_2 - x_2 y_1 ,$$

$$G_4 = e^{'2} z_1 G_3 .$$

Using this equations and the general formulae on the angle between vectors we will count the value of the azimuth from the following equation:

$$\cos(az) = \frac{G_1F_1 + G_2F_2}{\sqrt{F_1^2 + F_2^2}\sqrt{G_1^2 + G_2^2 + G_3^2}}. \tag{15}$$

The above expression is indeterminate for $x = y = 0$, in other words — on poles. Then F_1 and F_2 rise zero, when both the numerator and the denominator (15) acquires zero. However then it is clear that the azimuth is 180° or 0°.

Proposed method is also ineffective, when points P_1, P_2 and k are lying on straight line. Such situation takes place when both start point as well as final point are situated on the opposite poles (what was considered above), or when both points are on the equator, and their longitudes differ exactly by 180°. In such case the azimuth is equal to 270° or 90°.

Presented method of calculations of the azimuth was verified by the author by the comparison with performance of calculations presented in manuals of the geodesy. Coordinates of both points and the azimuth as well as the distance taken from the books are presented in the table 1. Applied method of calculations is given in the third column. In the last column the results of calculations with the new method are presented.

Table 1. Comparison of data taken from geodesy handbooks and results of azimuth calculation by new method [own study]

No.	Data	Method	Source	Results
1	B_1 = 54°22'17.2318"N L_1 = 018°46'49.0445"E B_2 = 62°41'36.8880"N L_2 = 002°44'58.4200"W D = 1 547 246m **Az = 315°21'24.876"**	Bessel's formulae	Czapczyk, Urbański, 1988	**Az = 315°21'25.9"**
2	B_1 = 54°22'17.2318"N L_1 = 018°46'49.0445"E B_2 = 57°00'47.6200"N L_2 = 013°49'19.5766"E D = 428 452m **Az = 315°21'23.8125"**	Gauss's mid-latitude formulae	Czapczyk, Urbański, 1988	**Az = 315°21'24.6"**
3	B_1 = 53°41'49.6935"N L_1 = 020°58'49.8323"E B_2 = 53°24'53.7999"N L_2 = 021°01'43.2998"E D = 31 569.5m **Az = 174°10'30.0120"**	Gauss's mid-latitude formulae	Dyrda, 1984	**Az = 174°10'30"**

No.	Data	Method	Source	Results
4	B_1 = 49°56’09.3536”N L_1 = 038°01’02.7546”E B_2 = 49°46’35.2034”N L_2 = 037°43’00.3488”E D = 27 967m **Az = 230°44’50.959”**	Clark’s best formulae	Hlibowicki, 1982	**Az = 230°44’51”**
5	B_1 = 54°12’35.0000”N L_1 = 018°33’15.0000”E B_2 = 55°05’48.6500”N L_2 = 018°54’07.5600”E D = 101 275m **Az = 012°40’12.1”**	Schreiber’s formulae	Banachowicz, Urbański, 1988	**Az = 012°40’12”**

CONCLUSIONS

In author’s opinion method proposed here will not replace well-known methods when calculations are performed ‘by hand’. However, in case of the computer calculations the Cartesian coordinate system possesses indubitable advantages. Many formulas in this coordinate system are simpler, and calculations do not demand usage of trigonometric functions, what consequently shortens the software and permits to avoid numeric difficulties. Examples presented in the Tab. 1 shows, that only on the distances longer 1 km the differences are bigger that 1”, so from point of view of navigator proposed method permits to obtain similar accuracy of calculations as precise well-known geodesic methods.

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