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THE EFFECT OF MATRIX ELEMENTS OF SYSTEM GEOMETRY ON THE ACCURACY OF THE HYPERBOLIC NAVIGATIONAL SYSTEM'S POSITION COORDINATES

ABSTRACT – The article presents formulas describing the values of average circular error, the average coordinate values and their covariance in the function of gradient matrix elements of hyperbolic position lines. The classical formulas do not take into account the correlation existing between the measurements of navigational parameters, and therefore also the correlation existing between position coordinates. This is also why they do not fully describe the accuracy of determined position coordinates. This is particularly significant in the case of strong correlation between coordinates and in cases when we are interested in directional error.

INTRODUCTION

In accuracy analysis of position coordinates the average circular error or the circular error with 95% probability are usually applied. The classical method of deriving formulas to the value of average circular error of position coordinates is based on errors of position line vectors [Bowditch, 1995], [Urbański et al., 1976]. From the point of view of contemporary navigational technique – automated navigational systems and microprocessor systems – this method is ineffective and requires additional calculations. It is better to make use of data already existing when calculating the position coordinates. This approach has been suggested in the present paper.

This article presents formulas describing the dependence of average circular error, average errors of position coordinates and the covariance of coordinates in the function of gradient matrix elements of hyperbolic position lines.
These dependences permit to take into account correlation in the measurement of navigational parameters, and thereby also the correlation of coordinate errors of position plotted.

These formulas can be applied both in the algorithm of integrated navigational systems and in predicting the accuracy of navigational position systems applied (or designed) and in reproducing all accuracy parameters of position coordinates on the basis of archive data (for instance: a full accuracy estimation of earlier hydrographic surveys, while analyzing navigational averages etc.).

AVERAGE POSITION CIRCULAR ERROR

Let us start with a general situation, when the hyperbolic system consists of four radionavigational stations. This possibility practically exists in LORAN-C, Raydist, a variety of BRAS and other short-range hyperbolic systems. This variant is illustrated by Fig.1. In this case the hyperbolic gradient matrix of position lines has the following form [Banachowicz, Urbański, 1988]:

$$\mathbf{G} = \begin{bmatrix} \cos A_i - \cos A_j & \sin A_i - \sin A_j \\ \cos A_l - \cos A_k & \sin A_l - \sin A_k \end{bmatrix}, \tag{1}$$

where A_i – the azimuth on the i-th radionavigational station. Taking into consideration what follows below, the simple trigonometric identities for the difference between the sines and cosines of two angles:

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

and

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

we will obtain another form of this matrix:

$$\mathbf{G} = \begin{bmatrix} -2\sin A_{ij}\sin\frac{\omega_{ij}}{2} & 2\cos A_{ij}\sin\frac{\omega_{ij}}{2} \\ -2\sin A_{lk}\sin\frac{\omega_{lk}}{2} & 2\cos A_{lk}\sin\frac{\omega_{lk}}{2} \end{bmatrix}, \tag{1'}$$

where:

A_{ij} – average azimuth between the i-th and the j-th station,

ω_{ij} – base angle between the i-th and the j-th station

In this figure P denotes the observer's point (the position of the radionavigational system's receiver antenna), and S_i – the position of the i-th system station. Other designations are the same as in the above formulas.

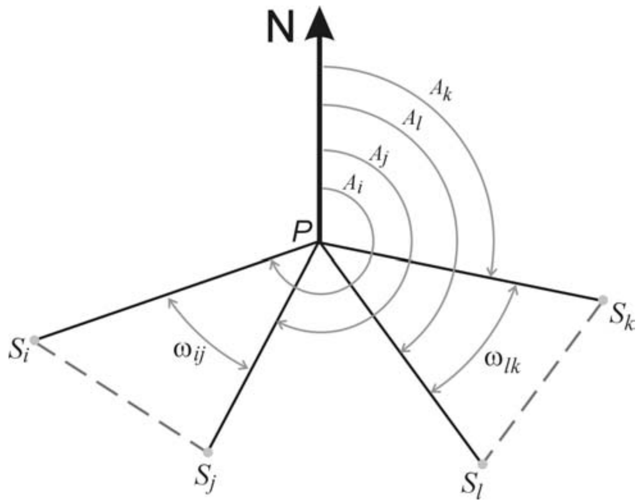

Fig. 1. Configuration of four radionavigational stations of the hyperbolic system

In accordance with formula (1'), the matrix is transposed in the following form:

$$\mathbf{G}^{\mathbf{T}} = \begin{bmatrix} -2\sin A_{ij}\sin\frac{\omega_{ij}}{2} & -2\sin A_{lk}\sin\frac{\omega_{lk}}{2} \\ 2\cos A_{ij}\sin\frac{\omega_{ij}}{2} & 2\cos A_{lk}\sin\frac{\omega_{lk}}{2} \end{bmatrix} \tag{2}$$

This is why the matrix of the positional system geometry $\mathbf{\Gamma}$ [Banachowic, 1991] will be equal to

$$\mathbf{\Gamma} = \mathbf{G}^{\mathbf{T}}\mathbf{G} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \tag{3}$$

where the particular elements of this matrix are described by the following formulas (it is a symmetric matrix, which is why the equation $\gamma_{12} = \gamma_{21}$ takes place):

$$\gamma_{11} = 4\left(\sin^2 A_{ij}\sin^2\frac{\omega_{ij}}{2} + \sin^2 A_{lk}\sin^2\frac{\omega_{lk}}{2}\right),$$

$$\gamma_{12} = \gamma_{21} = -4\left(\sin A_{ij} \cos A_{ij} \sin^2 \frac{\omega_{ij}}{2} + \sin A_{lk} \cos A_{lk} \sin^2 \frac{\omega_{lk}}{2} \right)$$

$$\gamma_{22} = 4\left(\cos^2 A_{ij} \sin^2 \frac{\omega_{ij}}{2} + \cos^2 A_{lk} \sin^2 \frac{\omega_{lk}}{2} \right).$$

The average circular error of position coordinates can be defined by the following equivalent formula [Banachowicz, 1991], [Banachowicz, 1993]:

$$M = \sigma_{\Delta D} \sqrt{tr\Gamma^{-1}}\ , \tag{4}$$

where:

$\sigma_{\Delta D}$ – measurement error of distance difference,
tr – denotes the matrix trace.

In effect, after calculating the matrix reverse to matrix (3) and further transformations, we will obtain the final form of the formula for the average position circular error

$$M = 0{,}5\sigma_{\Delta D} \operatorname{cosec}(A_{ij} - A_{lk})\sqrt{\operatorname{cosec}^2 \frac{\omega_{ij}}{2} + \operatorname{cosec}^2 \frac{\omega_{lk}}{2}}\ . \tag{5}$$

Yet, hyperbolic navigational systems usually consist of three radionavigational systems (minimal number of position lines – two hyperbolas). This situation is pictured on the next figure (Fig.2). In this case, the middle station is common for both position lines, which means that the equation $j = l$ takes place. Then, the difference between average azimuths is equal to the angle of intersection of the position lines, i.e.:

$$\theta = A_{ij} - A_{jk}\ , \tag{6}$$

and the base angles will obtain the following indexes (markings)

$$\omega_{ij} = \omega_{12} \text{ and } \omega_{jk} = \omega_{23}\ . \tag{7}$$

Finally, we will have the following formula for the average circular error of position

$$M = 0{,}5\sigma_{\Delta D} \operatorname{cosec}\theta \sqrt{\operatorname{cosec}^2 \frac{\omega_{12}}{2} + \operatorname{cosec}^2 \frac{\omega_{23}}{2}}\ , \tag{8}$$

which is in accordance with the classic formulas [Banachowicz, 2001], [Urbański et al., 1976].

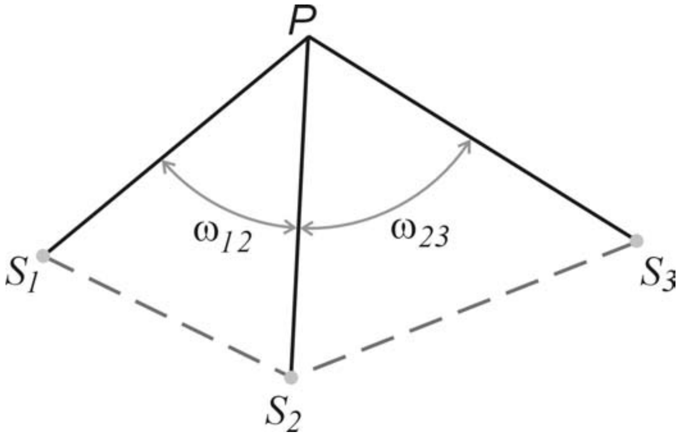

Fig. 2. Configuration of three radionavigational stations of the hyperbolic system

AVERAGE ERRORS OF COORDINATES AND COVARIANCE BETWEEN COORDINATES

Making use also of the other elements of the system Γ geometry matrix we can calculate the average errors of the coordinates being determined and the covariance between them, that is all elements of normal distribution of a two-dimensional random variable. These characteristics permit a fuller analysis of the position coordinates being determined and the estimation of accuracy in a specified direction [Banachowicz, 2001]. This is essentially significant when navigating in difficult areas, e.g. on fairways, or in the case of determining linear values (distances between points are calculated on the basis of coordinates, the isobath direction, the course of the fairway edge etc.)

The particular average errors and covariance are obtained by using the definition of the position covariance matrix [Banachowicz, 1991]:

$$\mathbf{P} = \begin{bmatrix} \sigma_\varphi^2 & \sigma_{\varphi\lambda} \\ \sigma_{\varphi\lambda} & \sigma_\lambda^2 \end{bmatrix} = \sigma_{\Delta D}^2 \Gamma^{-1} \tag{9}$$

Due to the limited size of this paper we present the final formulas right now, passing over the algebraic transformations.

a) Average error of geographic latitude determination

$$\sigma_\varphi = 0{,}5\sigma_{\Delta D} \operatorname{cosec}(A_{ij} - A_{lk})\sqrt{\cos^2 A_{ij} \operatorname{cosec}^2 \frac{\omega_{lk}}{2} + \cos^2 A_{lk} \operatorname{cosec}^2 \frac{\omega_{ij}}{2}} \tag{10}$$

and for the common middle station

$$\sigma_{\varphi} = 0{,}5\sigma_{\Delta D} \operatorname{cosec}\theta \sqrt{\cos^2 A_{12} \operatorname{cosec}^2 \frac{\omega_{23}}{2} + \cos^2 A_{23} \operatorname{cosec}^2 \frac{\omega_{12}}{2}} \,. \quad (11)$$

b) Average error of geographic longitude determination

$$\sigma_{\lambda} = 0{,}5\sigma_{\Delta D} \operatorname{cosec}(A_{ij} - A_{lk}) \sqrt{\sin^2 A_{ij} \operatorname{cosec}^2 \frac{\omega_{lk}}{2} + \sin^2 A_{lk} \operatorname{cosec}^2 \frac{\omega_{ij}}{2}} \quad (12)$$

and for the common middle station

$$\sigma_{\lambda} = 0{,}5\sigma_{\Delta D} \operatorname{cosec}\theta \sqrt{\sin^2 A_{12} \operatorname{cosec}^2 \frac{\omega_{23}}{2} + \sin^2 A_{23} \operatorname{cosec}^2 \frac{\omega_{12}}{2}} \,. \quad (13)$$

c) Covariance between geographic coordinates:

$$\sigma_{\varphi\lambda} = \frac{1}{8}\sigma_{\Delta D}^2 \operatorname{cosec}^2 (A_{ij} - A_{lk}) \left(\sin 2A_{ij} \operatorname{cosec}^2 \frac{\omega_{lk}}{2} + \sin 2A_{lk} \operatorname{cosec}^2 \frac{\omega_{ij}}{2} \right) \quad (14)$$

and for the common middle station

$$\sigma_{\varphi\lambda} = \frac{1}{8}\sigma_{\Delta D}^2 \operatorname{cosec}^2 \theta \left(\sin(A_1 + A_2) \operatorname{cosec}^2 \frac{\omega_{23}}{2} + \sin(A_2 + A_3) \operatorname{cosec}^2 \frac{\omega_{12}}{2} \right). \quad (15)$$

Naturally, calculating the average circular error using dependencies (11) and (13) we will also obtain formula (8). But now, additionally taking into consideration dependence (15) we can calculate the elements of average error ellipsis, which cannot be done when using classical formulas. The elements of average error ellipsis are calculated on the basis of average errors of position coordinates and the covariance between coordinates using the formulas below [Banachowicz, 1991]. These formulas are the solution of the characteristic equation of covariance matrix P.

The square of the semi-major axis is expressed by the following dependence:

$$a^2 = \frac{1}{2}\left(\sigma_{\varphi}^2 + \sigma_{\lambda}^2 + \sqrt{\left(\sigma_{\varphi}^2 - \sigma_{\lambda}^2\right)^2 + 4\sigma_{\varphi\lambda}^2} \right). \quad (16)$$

While the square of the semi-minor axis is described by the formula:

$$b^2 = \frac{1}{2}\left(\sigma_{\varphi}^2 + \sigma_{\lambda}^2 - \sqrt{\left(\sigma_{\varphi}^2 - \sigma_{\lambda}^2\right)^2 + 4\sigma_{\varphi\lambda}^2} \right). \quad (17)$$

Orientation of the average error ellipsis is defined by the following dependence:

$$\alpha = \frac{1}{2} \operatorname{arctg} \frac{2\sigma_{\varphi\lambda}}{\sigma_{\varphi}^2 - \sigma_{\lambda}^2} \,. \quad (18)$$

The centre of error ellipsis is determined by the average values of coordinates $\overline{\varphi}$ and $\overline{\lambda}$, while the formulas (10) – (15) describe the other elements of probability density function of the two-dimensional random variable.

CONCLUSION

The dependences given above permit a full accuracy estimation of position coordinates determined by the hyperbolic position system – the calculation of particular coordinates' errors, their correlation, elements of the average error ellipsis and the circular errors. This method can be generalized for any position lines. It can also be adapted to calculate and analyze the accuracy of various, unequally precise and correlated measurements of navigational parameters.

It also permits to analyze the accuracy of archival data – the course of the voyage, situations of average or previously carried out hydrographic surveys. These surveys should be supplemented by a description of the position system configuration and the average errors of phase measurements (differences of distance). Then, all the accuracy indexes of position coordinates can be reproduced at any moment. This provides wide possibilities of using archival surveys to update nautical information and to identify objects by comparing their average values of position coordinates, and to make a full estimation of accuracy (testing of averages).

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